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Reconstruction of Quantum Theory

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ABSTRACT

What belongs to quantum theory is no more than what is needed for its derivation. Keeping to this maxim, we record a paradigmatic shift in the foundations of quantum mechanics, where the focus has recently moved from interpreting to reconstructing quantum theory. Several historic and contemporary reconstructions are analyzed, including the work of Hardy, Rovelli, and Clifton, Bub and Halvorson. We conclude by discussing the importance of a novel concept of intentionally incomplete reconstruction.

1 What is Wrong with Interpreting Quantum Mechanics

Ever since the first days of quantum mechanics physicists as well as philosophers have tried to interpret it, understanding this task as a problem of giving to the new physical theory a clear meaning. A principal reason for why one always felt a need for interpretation has to do with the puzzling aspect of the formalism of quantum mechanics known as the measurement problem. Reversible unitary evolution of the wave function is at the moment of measurement replaced, according to standard quantum mechanics, by an

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irreversible transformation referred to as wavefunction collapse. First and foremost, interpretations of quantum mechanics aimed at making sense of this surprising change in the theory's dynamics, sometimes taking the collapse at face value and claiming its fundamental irreducible role, or sometimes going to another extreme and denying the collapse altogether. However, looking globally, the enterprise of interpreting quantum mechanics has failed: today we still have no consensus on what *the* meaning of quantum theory is. None of the proposed answers has won overall acceptance. Perhaps the most remarkable manifestation of the failure to interpret quantum mechanics is the attitude taught to most young physicists in lecture rooms and research laboratories in the last half century, 'Shut up and calculate!' (Mermin [2004]).

Why did attempts at a univocal interpretation fail? Many answers are possible, and among them we favour two, both showing that there is an intrinsic deficiency in the idea of interpreting a physical theory only by using philosophical instruments.

The first answer is that to a physical theory one would naturally like to give a *physical* meaning in the Greek sense of $\varphi\acute{\upsilon}\sigma\iota\varsigma$, that is, we—as part of the physicists' audience—expect to be told a true story about nature. This is because we casually tend to apply physical theory to the phenomenal world to learn something about the latter, and not the world to physical theory, in order to invent a meaning of the theory. Physical theory is, above all, a tool for predicting the yet unobserved phenomena; so employing existing knowledge and experience of the world to interpret physics runs counter to its basic function as scientific theory. Still, in spite of such an against-the-grain direction followed by a philosophical interpretation, this does not necessarily lead to formal contradiction that could invalidate the interpretation program logically; more modestly, but perhaps no less irritatingly, at the end one is often left with a feeling of being excluded from mainstream research. Further, as the physics of today is inseparable from mathematics, a meaning cannot be physical and thus satisfactory if it is merely heaped over and above the mathematical formalism of quantum mechanics, instead of coming all the way along with the formalism as it rises in a derivation of the theory.

The second answer is that we live in a situation where objective truth has been appropriated by science, and to pass public ratification every increase in knowledge must confront experimental setups. In this world an interpretation can only be considered satisfactory when it becomes an integral part of science. This is not unprecedented in the history of ideas: indeed, with the advent of empirical science many philosophical questions ceased to be taken as philosophical and became subject to the scientific inquiry. Even pronounced critics of the idea that physics can have implications for philosophy admit occasionally that physics and philosophy can be connected, for example, in that 'discoveries in physics sometimes reveal that topics that had been thought to be proper subjects for philosophical argument actually belong in the province of ordinary science' (Weinberg [1996]). To be able to convince the skeptics in the philosophical debate over interpretations of quantum mechanics, the problem of interpreting quantum mechanics must be treated as science to a highest possible degree. Only then will the puzzling discord disappear.

2 Reconstruction of Physical Theory

2.1 Schema

We call *reconstruction* the following schema adopted to the needs of quantum theory and different from the notion of rational reconstruction, introduced by Carnap ([1928]): Theorems and major results of physical theory are formally derived from simpler mathematical assumptions. These assumptions or axioms, in turn, appear as a representation in the formal language of a set of physical principles. Reconstruction consists therefore of three stages: first, give a set of physical principles, then formulate their mathematical representation, and finally rigorously derive the formalism of the theory.

Contrary to interpretation, the three-stage structure of reconstruction permits the latter to acquire supplementary persuasive power, due to the use of mathematical derivation. Established as valid results, theorems and equations of the theory become unquestionable and free of suspicion. 'Why is it so?'—'Because we derived it.' The question of meaning, previously asked with regard to the formalism, is removed and now bears, if at all, only on the selection of the principles. No room for mystery remains in what concerns the meaning of the theory's mathematical apparatus. This implies, among other things to be discussed later, that in the reconstruction program the measurement problem loses the central role it has occupied for the success of an interpretation of quantum mechanics. One now makes sense of all of the formalism solely on the basis of the first principles, and whatever mathematical element is contained in the formalism that is used in a particular reconstruction, it now acquires a precise meaning in virtue of the first principles.

Explanatory power of the reconstruction is a power of explanation of where the structure of the theory comes from; not necessarily a power of explanation by the theory, of the real world. Traditionally, interpretations have been paying attention to the latter task and giving less importance to the former. Reconstruction shifts one's focus: its added value for better understanding quantum theory originates in the new insights into the structure of the theory, made possible thanks to the use of mathematical derivation.

2.2 Selection of the first principles

Anyone who wishes to attempt a reconstruction of physical theory must formulate the foundational principles which he or she believes plausible and translate them into mathematical axioms. Then the rest of the theory will be constructed 'mechanically,' by means of a formal derivation. The choice of axioms must be the only allowed freedom in the whole construction. It is commonplace to say that it is not easy to exhibit an axiomatic system that would stand up to such requirements, especially in the case of quantum theory.

First, where do candidate axioms for quantum theory come from and how does one judge which statements can plausibly be taken as axioms? Prior to pronouncing such a judgment, one must develop an intuition of what is plausible about quantum theory and what is not. This can only be achieved by *practicing* the theory, that is, by taking its prescriptions at face value, applying them to systems under consideration in particular tasks, and obtaining results. In short, one needs to acquire a real 'know-how' above and beyond a purely theoretical knowledge that quantum mechanics could solve such or such other problems. The scientist's intuition develops from experience—it cannot arise from the abstract, *in principle* knowledge.

Still, taking prescriptions of quantum theory at face value, applying them, and obtaining results, will not yet make things *clear* about quantum mechanics. Indeed, one can possess knowledge about how to apply a certain tool, without caring about the structure of the tool nor its meaning. The quantum mechanical know-how serves purely as such a tool for developing one's intuition about which candidate idea would be a plausible foundational principle, and which other candidate idea will not pass the test. Candidate foundational principles need not even be theorems of the already existing quantum mechanics: one's judgment may be such that a new statement—false or only conditionally true in quantum theory—will be taken as axiom in reconstruction of a new theory. Examples of such principles will be discussed in Section 3.5.

Second, what should one require from the first principles? They must be simple *physical statements*, that is assertions, such that their meaning is immediately, easily accessible to a scientist's understanding. They must also, although this can only be used as an *a posteriori* criterion, allow for a clear and unambiguous translation of their content into mathematically formulated axioms. Derivation of quantum theory will then rely on these axioms.

2.3 Status of the first principles

The reconstruction program includes a *derivation* of quantum theory, but in the previous section one was told to *apply* and *use* it in order to motivate the derivation. Is there a vicious circle here? We submit that there is none, and this is thanks to the status of the first principles: namely, they must not be necessarily taken as ultimate truths about nature. Independently of one's ontological commitments, the first principles have only a minimal epistemic status of being postulated for the purpose of reconstructing the theory in question. As in the 19th-century mathematics, in theoretical physics the axiomatic method is to be separated from the attitude that the Greeks had toward axioms: that they represent the truth about reality. Much of the progress of mathematics is due to understanding that an axiom may no longer be considered an ultimate truth, but merely a fundamental structural element, that is, an assumption that lies at the basis of a certain theoretical structure. In mathematics, after departing from the Greek concept of axiom, 'not only geometry, but many other, even very abstract, theories have been axiomatized, and the axiomatic method has become a powerful tool for mathematical research, as well as a means of organizing the immense field of mathematical knowledge which thereby can be made more surveyable' (Heyting [1963]). A similar attitude is to be taken with respect to axioms used for the formal derivation of a physical theory. To summarize, the methodological precept that gives a minimal status to the first principles in a reconstruction program, runs as follows:

- If the theory itself does not tell you that the states of the system, or any other variables, are ontic, then do not take them to be ontic.

To understand this precept, return first to the idea that in developing an intuition with respect to the plausibility of the foundational principles used to derive a theory, one takes this theory as a given and applies it practically, so as to acquire a know-how that would justify the choice of principles. Now, when one is working with several physical theories, ideas that have previously served as foundations for theory I, may turn out to be derivative (i.e., theorems) in theory II. Examples include the case of thermodynamics and statistical physics, or the relation between macrolevel hydrodynamics and low-level molecular theory of liquids. This demonstrates the limits of philosophical assumptions that can be made with respect to the status of the first principles used in reconstruction of a given physical theory. Indeed, nothing can be generally said about their ontological content or the ontic commitments that arise from these principles. It is more economical, and amounts to a certain *epistemological modesty*, to treat the foundational principles as axioms *hic et nunc*, that is, in a given theoretical description. Epistemological modesty requires that one brackets his or her personal motives for the choice of first principles, and that one reconstructs the theory based solely on the principles themselves. Reconstruction becomes meaningful due to the content of the first principles on which it relies.

Reconstruction of a physical theory has its main advantage compared to philosophical interpretation of the theory in the fact that it moves a number of questions, previously thought of as philosophical, to the realm of science, and this in virtue of the mathematical derivation which the reconstruction program operates. However, philosophical problems do not altogether disappear; they still apply to the first principles and take the form of a problem of their *justification*. Evidently, it is a minimal logical condition that such a justification must not be seen as mathematical deduction of the principles from the theory in whose very foundations they lie. Once one obtains a full formalism of the theory in an epistemologically modest reconstruction, it is then possible to ask the reconstructed theory itself if it allows a realist interpretation of the first principles from which it has been derived, or, perhaps, it imposes constraints on such possible ontological commitments. While, in general, the status of the first principles as ultimate truths about reality is not a necessity, certain reconstruction programs permit that this status be safely, or almost, attributed to their first principles. The task of justification is therefore external to the reconstruction program and must be executed by one operating under a different set of assumptions, that is, by taking the theory as a given, and motivating from this standpoint why the principles that have been involved in the reconstruction are simple, physical, and plausible. Thus, philosophy is not fully chased out of physics. By demarcating the frontier between what can be treated as a scientific question and what belongs to the metalevel of analysis, one contributes to a better understanding of the structure of the theory and of those of its foundational postulates that require a metatheoretic interpretation and justification.

It is often claimed that applying to theoretical physics the same methodology as in axiomatizations in mathematics leads to a problem, exposed by Einstein. While supporting the axiomatic move in mathematics in that it 'dispels the obscurity which formerly surrounded the principles,' Einstein argues that if one wants to apply a similar move in physics, then one has to face the difficulty of connecting 'conceptual schemata' with 'real objects' (Einstein [1921]). Applied to an epistemologically modest reconstruction, this problem is no more than apparent, if the status of the first principles is properly freed from ontological commitments. The latter do have a bearing indeed, but only on justifying the choice of postulates. Starting from a particular set of principles (stage 1) represented formally (stage 2), mathematical derivation (stage 3) proceeds in exactly the same way as in mathematics. Reconstruction understood as stages 1 – 3 is therefore analogous to an axiomatization in mathematics. Still, it differs from mathematical axiomatization in that it also invokes the problem of justification of the choice of first principles. However, when the first principles are formulated in an epistemologically modest way, Einstein's 'conceptual schemata,' or structural elements of physical theory, are its only building blocks. Unambiguous derivation of the theory's formalism is detached from the question of reality of the world that the theory describes, with respect to which one is free to take a variety of positions. For quantum theory, this detachment amounts to operating a reconstruction of quantum theory from a set of first principles devoid of the necessity of being justified on the ontological grounds.

3 Examples of Reconstruction

3.1 Early examples of reconstruction

In the last decade reconstruction became a major trend in the foundations of quantum mechanics. Before describing this recent work, let us first look further back in the history of quantum mechanics: there too axiomatic derivations occupy an eminent place. The first paper where quantum mechanics was treated axiomatically appeared shortly after the creation of quantum mechanics itself: in 1927 Hilbert, von Neumann and Nordheim stated their view of quantum mechanics as one in which '...[the theory's] analytical apparatus, and the arithmetic quantities occurring in it, receives *on the basis of the physical postulates* a physical interpretation. Here, the aim is to formulate the physical requirements so completely that the analytical apparatus is just uniquely determined. Thus the route is of axiomatization' (Hilbert *et al.* [1927], emphasis added). It is on this route of axiomatization that von Neumann in collaboration with Birkhoff was led to study the logic of quantum mechanics (Birkhoff and von Neumann [1936]). Following their work, many axiomatic systems were proposed, for example by Zieler ([1961]), Varadarajan ([1962], [1968]), Piron ([1964], [1972]), Kochen and Specker ([1965]), Guenin ([1966]), Gunson ([1967]), Jauch ([1968]), Pool ([1968a], [1968b]), Plymen ([1968b]), Marlow ([1978]), Beltrametti and Casinelli ([1981]), Holland ([1995]), or Ludwig ([1985]). Another branch of axiomatic quantum theory, the algebraic approach, was first conceived by Jordan *et al.* ([1934]) and later developed by Segal ([1947], [1963]), Haag and Kastler ([1964]), Plymen ([1968a]), Emch ([1972]) and others; for a recent review, see (Brunetti and Fredenhagen [unpublished]).

However, a vast majority of these axiomatic developments do not fall under our notion of reconstruction, as they were based on highly abstract mathematical assumptions and not, as we require, on simple physical principles. Consider for instance an exemplary work by Mackey ([1957], [1963]). He develops quantum mechanics as follows. Take a set $\mathcal{B}$ of all Borel subsets of the real line and suppose we are given two abstract sets $\mathcal{O}$ (a to-be space of observables) and $\mathcal{S}$ (a to-be space of states) and a (to-be probability) function P which assigns a real number $0 \leq P(x, f, M) \leq 1$ to each triple x, f, M, where x is in $\mathcal{O}$, f is in $\mathcal{S}$, and M is in $\mathcal{B}$. Assume certain properties of P listed in axioms M1–M9:

M1 Function P is a probability measure. Mathematically, we have $P(x, f, \emptyset) = 0$, $P(x, f, \mathbb{R}) = 1$, and $P(x, f, M_1 \cup M_2 \cup M_3 \ldots) = \sum_{n=1}^{\infty} P(x, f, M_n)$ whenever the M_n are Borel sets that are pairwise disjoint.

M2 Two states, in order to be different, must assign different probability distributions to at least one observable; and two observables, in order to be different, must have different probability distributions in at least one state. Mathematically, if $P(x, f, M) = P(x', f, M)$ for all f in $\mathcal{S}$ and all M in $\mathcal{B}$ then $x = x'$; and if $P(x, f, M) = P(x, f', M)$ for all x in $\mathcal{O}$ and all M in $\mathcal{B}$ then $f = f'$.

M3 Let x be any member of $\mathcal{O}$ and let u be any real bounded Borel function on the real line. Then there exists y in $\mathcal{O}$ such that $P(y, f, M) = P(x, f, u^{-1}(M))$ for all f in $\mathcal{S}$ and all M in $\mathcal{B}$.

M4 If $f_1, f_2, \ldots$ are members of $\mathcal{S}$ and $\lambda_1 + \lambda_2 + \ldots = 1$ where $0 \leq \lambda_n \leq 1$, then there exists f in $\mathcal{S}$ such that $P(x, f, M) = \sum_{n=1}^{\infty} \lambda_n P(x, f_n, M)$ for all x in $\mathcal{O}$ and M in $\mathcal{B}$.

M5 Call *question* an observable e in $\mathcal{O}$ such that $P(e, f, \{0, 1\}) = 1$ for all f in $\mathcal{S}$. Questions e and e' are disjoint if $e \leq 1 - e'$. Then a question $\sum_{n=1}^{\infty} e_n$ exists for any sequence (e_n) of questions such that e_m and e_n are disjoint whenever $n \neq m$.

M6 If E is any compact, question-valued measure then there exists an observable x in $\mathcal{O}$ such that $\chi_M(E) = E(M)$ for all M in $\mathcal{B}$, where χ_M is a characteristic function of M.

M7 The partially ordered set of all questions in quantum mechanics is isomorphic to the partially ordered set of all closed subspaces of a separable, infinite-dimensional Hilbert space.

M8 If e is any question different from 0 then there exists a state f in $\mathcal{S}$ such that $m_f(e) = 1$.

M9 For each sequence (f_n) of members of $\mathcal{S}$ and each sequence (λ_n) of non-negative real numbers whose sum is 1, one-parameter time evolution group $V_t : \mathcal{S} \mapsto \mathcal{S}$ acts as follows: $V_t\left(\sum_{n=1}^{\infty} \lambda_n f_n\right) = \sum_{n=1}^{\infty} \lambda_n V_t(f_n)$ for all $t \geq 0$; and for all x in $\mathcal{O}$, f in $\mathcal{S}$, and M in $\mathcal{B}$, $t \rightarrow P(x, V_t(f), M)$ is continuous.

In Mackey's nine axioms all essential features of the quantum formalism are directly postulated in their mathematical form: the Hilbert space structure in M5–M8, the state space and probabilistic interpretation in M1–M4, and the time evolution in M9. The list of axioms is long and their meaning far from transparent. Indeed, at no point is one given an intuition as to where these mathematical definitions come from or how one could justify them on physical rather than formal grounds. In fact, Mackey's concern in the early 1950s was with a precise mathematical axiomatization of quantum mechanics rather than with the question of what quantum mechanics tells us about the world or with reconstructing its formalism from the set of such fundamental ideas. Thus, the first stage of the reconstruction schema, at which one formulates physical principles, is absent from Mackey's work, and instead one starts directly at the second stage, where the first principles appear in mathematical form.

Mackey's axioms M5–M8 were consequently reformulated in the language of quantum logic, thereby rephrasing the assumptions that underlie the Hilbert space structure. This has been the case, most prominently, in (Jauch [1968]; Piron [1964], [1972]), and also in an important state-of-the-art book (Beltrametti and Cassinelli [1981]). These quantum logical assumptions are simple enough to be accessible for direct comprehension, in contrast to Mackey's mathematically formulated axioms, but they tend to be linguistic rather than physical. This means that one typically argues that it makes no sense to *speak* about certain concepts unless some suitable 'trivial' properties of these concepts had been postulated, for example, the notion of proposition is only meaningful if, as in (Demopoulos [2004]), negation or partial order, or, as in (Beltrametti and Cassinelli [1981]), implication, are defined. Although we fully acknowledge that linguistic *a priori* arguments can be interesting and powerful, we however distinguish them from the reconstruction program: in the latter, first principles from which the theory is derived have a *physical* meaning, that is, they tell us something directly and intuitively comprehensible about the world. Such principles must be independent of a particular formalism that one further employs to derive quantum theory, and therefore must not rely on quantum logic as just one among many such formalisms.

3.2 Hardy's reconstruction

It is startling how Mackey's and similar axiomatic sets for quantum mechanics differ from systems of first principles proposed by several contemporary authors. Although some of these systems remain very much in the spirit of earlier proposals of sets of abstract mathematical postulates (e.g., Pitowsky [2006]), even in such cases the author typically feels the need to give a non-technical, physical motivation for the choice of axioms. Still further on the way to foundational physical principles rather than purely mathematical axioms, one finds an interesting example of reconstruction coming from Hardy's instrumentalist derivation of quantum theory (Hardy [unpublished (b)]). Unlike Mackey, who starts with two large abstract sets and an abstract real-valued function, Hardy's 'five reasonable axioms' set up a link between two initially introduced natural numbers, K and N. K is the number of degrees of freedom of the system and is defined as the minimum number of probability measurements needed to determine the state. Dimension N is defined as the maximum number of states that can be reliably distinguished from one another in a single measurement. The axioms are:

H1 *Probabilities.* In the limit as n becomes infinite, relative frequencies (measured by taking the proportion of times a particular outcome is observed) tend to the same value for any case where a given measurement is performed on an ensemble of n systems prepared by some given preparation.

H2 *Simplicity.* K is determined by a function of N where $N = 1, 2, \ldots$ and where, for each given N, K takes the minimum value consistent with the axioms.

H3 *Subspaces.* A system whose state is constrained to belong to an M dimensional subspace behaves like a system of dimension M.

H4 *Composite systems.* A composite system consisting of subsystems A and B satisfies $N = N_A N_B$ and $K = K_A K_B$.

H5 *Continuity.* There exists a continuous reversible transformation on a system between any two pure states of that system.

Hardy's list of axioms is considerably shorter and simpler than Mackey's, and although four of H1–H5 still use mathematical language in their formulation, the meaning of the axioms in Hardy's instrumentalist setting can be grasped easier than the meaning of Mackey's M1–M9. In fact, this meaning is already suggested by the names given to the axioms by Hardy. One can rephrase H1–H4 into *physical principles* from which one derives the formalism of the theory. This would provide for the missing first stage of the reconstruction schema and thus amount to a complete example of a reconstruction. Thus, at the first stage of reconstruction, the following physical principles are postulated that rephrase Hardy's axioms; they are simple but non-trivial: for H1, assume that probability can be introduced as *relative frequency* and it is a well-defined concept obeying the laws of probability theory; for H2, assume that the number of parameters needed to characterize a state is linked in a minimal way to the number of states that can be distinguished in one measurement, that is, information carrying capacity of the system; for H3, that systems that have the same information carrying capacity have the same properties; for H4, assume multiplicability of the information carrying capacity. At the second stage of the reconstruction, one formulates these principles mathematically, for example, in Hardy's fashion; at the third stage, one uses Hardy's theorems to derive the full-blown formalism of quantum mechanics.

A particular instrumental philosophy does not play a crucial role in the derivation: Hardy himself acknowledges that his axioms can be adopted by a realist as well as a hidden variable theorist or a partisan of collapse interpretations. Thus, the choice of the underlying philosophy is not critical to the success of the derivation, and Hardy's reconstruction advances our understanding of quantum theory irrespectively of the justification which one may have for the axioms. What matters are the simple physical principles formulated as axioms H1–H4. This is exactly what one would expect given the status of the first principles. We shall however see below an opposite example, where the justification used for the fundamental principles will limit the applicability and meaningfulness of the mathematical derivation.

Still, in Hardy's case it is not so clear whether axiom H5 has a *physical* meaning. Because it is this axiom that makes the theory quantum rather than classical, the reconstruction program cannot be said to be completely implemented. To illustrate this point, we distinguish two types of continuity assumptions that are made in axiomatic derivations of quantum theory. Continuity assumptions of type 1 select the correct type of the numeric field that is used in the construction of the Hilbert space of the theory: namely, of the field $\mathbb{C}$ of complex numbers. Solèr's theorem (Solèr [1995]) or Zieler's axioms (Zieler [1961]) are examples of type 1 continuity assumptions. Hardy's case is different and is an example of a continuity assumption of type 2, which is ultimately responsible for the appearance of the superposition principle. Examples of other type 2 assumptions include Gleason's ([1967]) non-contextuality, Brukner and Zeilinger's ([2003]) homogeneity of parameter space, Landsman's ([1998]) two-sphere property, or Holland's ([1995]) axioms C and D which bear a particular resemblance to Hardy's H5:

(C) Superposition principle for pure states:
  1. Given two different pure states (atoms) a and b, there is at least one other pure state c, $c \neq a$ and $c \neq b$ that is a superposition of a and b.
  2. If the pure state c is a superposition of the distinct pure states a and b, then a is a superposition of b and c.

(D) Ample unitary group: Given any two orthogonal pure states $a, b \in \mathcal{L}$, there is a unitary operator U such that $U(a) = b$.

We see that various axiomatic systems for quantum theory contain, under one form or another, the assumption of continuity, and it is this assumption which is largely responsible for making things quantum. Whatever the framework of the reconstruction program, bringing in topological considerations is essential. As it is exceedingly difficult to formulate a physical principle which may provide a meaning for the continuity assumptions of type 2, all reconstruction programs that employ them suffer from intrusion of an element of mathematical abstraction.

In a more recent reformulation of his axioms Hardy ([unpublished (c)]; private communication, April 2006) suggests a way to avoid the mathematically abstract type 2 continuity assumption that he has previously made in axiom H5. In the new version Hardy maintains H5 without the word ‘continuous’:

H5′ *Reversibility*. There exists a reversible transformation on a system between any two pure states of that system.

It is then hypothesized that this new reversibility axiom, which is strictly weaker than the former H5, if combined together with axiom H3 about the structure of subspaces, will allow one to derive continuity in any finite-dimensional space. The idea is to build on the fact that a limited number of finite groups is available in low-dimensional spaces. If the precision of a reversible transformation between pure states in H5′ is sufficiently small, then this transformation, by virtue of being an element of the group of reversible transformations, will necessarily be continuous, because no finite group will be available for this transformation to belong to, with a number of elements required for the carefully adjusted precision. If further research shows that this conjecture can be carried through and turned into a theorem, then for the first time one will have a set of algebraic assumptions H1–H5′ that could be used to reconstruct either classical or quantum theory, and the latter will be selected by fixing a particular value of the precision parameter in reversible transformations between pure states. Solèr demonstrated that the topology of quantum theory can be obtained from a set of algebraically formulated assumptions, in the case of infinite-dimensional space (Solèr [1995]). Hardy’s conjecture, if proved, will show how to achieve this derivation of topology from algebra in the case of finite-dimensional spaces. Astonishingly, in this conjectured reconstruction program the choice between classical and quantum theory would only depend on the value of one numeric parameter.

3.3 Rovelli’s reconstruction

The critique expressed in Section 3.2 with respect to Hardy’s continuity axiom applies to the example of reconstruction initially proposed by Rovelli ([1996]), that I have developed elsewhere (Grinbaum [2003], [2005]). Here, the reconstruction starts from two information-theoretic axioms:

R1 There exists a maximum amount of relevant information that can be extracted from a system.

R2 It is always possible to obtain new information about the system.

It may seem that R1 and R2 contradict each other. Indeed, R1 says that the quantity of information is finite, while from R2 it follows that this quantity must be infinite, because it is always possible to obtain some new information. The reason why there is no contradiction lies in the use of the term 'relevant' in R1, that does not appear in R2. Relevance of new information must be judged with respect to the information already possessed by the observer. Bringing about new information can not only increase the amount of information currently available to the observer, but also reduce it, due to the fact that some previously relevant information may become irrelevant. Thus, what is relevant depends on a particular sequence of questions asked by the observer, and for different observers the quantitative change in the amount of information brought in by the seemingly identical new fact may also differ.

Facts are only defined relative to an observer. Information too, in Rovelli's reconstruction, is observer-dependent. This notion corresponds to information in Shannon's sense, which is indexed by two variables: first pointing at the observed system, the one that the obtained information is about, and second pointing at the observer. It is impossible, in Rovelli's view, to separate the notion of information from its second index and to speak about objective information independently of a particular observer.

From R1 and R2, one derives the formalism of quantum mechanics using several auxiliary quantum logical assumptions. If one postulates that information is obtained through answers to yes-no questions that can be asked about the system, then the auxiliary assumptions are that the set of such yes-no questions forms a complete atomic orthocomplemented lattice. From axiom R1 and a formal definition of relevance of yes-no questions with respect to each other, one derives that this lattice is orthomodular. If a further assumption is made about the lattice of questions being isomorphic to the lattice of all closed subspaces of a Banach space constructed over a numeric field (i.e., real or complex numbers or quaternions), one then obtains that this Banach space is indeed a Hilbert space. Its quantum rather than classical character follows from axiom R2.

While the auxiliary assumptions cast a shadow on the conceptual clarity of the reconstruction much in the same fashion as does axiom H5 for Hardy's approach, the whole program presents itself differently from Hardy's instrumentalism. Mathematical derivation being still devoid of ontological commitments, the proposed justification of the first principles does not refer to an ontology. Rather, by reconstructing quantum theory from information-theoretic principles, we point at its epistemological character and at its role as a theory of (a certain kind of) knowledge, one with certain limits on the kind of information one may be dealing with. The most general theory of this kind of information takes the form of quantum theory. Here again reconstruction appears more appealing than a mere interpretation as it leaves room for any justification of first principles, some such justifications being possibly different from ours. Indeed, one may equally well choose to adopt a specific ontological picture to justify R1–R2. At the same time, regardless of a concrete philosophical justification for first principles, the meaning of quantum theory stands clear: it is a general theory of information constrained by several information-theoretic principles.

3.4 The CBH reconstruction

Clifton, Bub, and Halvorson ([2003]) propose a set of quantum informational constraints (referred to as 'CBH axioms') from which one derives the basic elements of quantum theory. They postulate three fundamental principles:

CBH1 No superluminal information transfer via measurement.

CBH2 No broadcasting.

CBH3 No bit commitment.

CBH argue that the meaning of axiom CBH1 is that when Alice and Bob perform local measurements, Alice's measurements can have no influence on the statistics for the outcomes of Bob's measurements, and vice versa. They also submit that 'otherwise this would mean instantaneous information transfer between Alice and Bob' and 'the mere performance of a local measurement (in the nonselective sense) cannot, in and of itself, transfer information to a physically distinct system.' These claims imply that CBH take the terms *distinct* and *distant* to be synonyms. Such an identification might indeed be a tacit assumption among quantum information theorists, whose theoretical needs do not exceed the case of finite-dimensional Hilbert spaces; but in the full-blown C^*-algebraic framework chosen by CBH, the meaning of the two terms is quite different. One has here an example of the way in which the initial quantum informational departure point of the CBH1–CBH3 principles constrains the use of the C^*-algebraic formalism only to situations where these principles make sense from the point of view of quantum information; in fact, the formalism is at the same time routinely applied to other settings as well. Unlike Hardy's derivation, which is independent of the particular instrumental justification of its first principles, the CBH reconstruction cannot be carried through outside the field of quantum information, because its mathematics, while still valid outside this field, requires a new justificatory language. Besides the problem of synonymy of 'distant' and 'distinct,' the quantum informational departure point also restricts the question of time evolution. The latter is tacitly taken by CBH to be the usual quantum mechanical time evolution, while in the general C^*-algebraic framework this is typically not the case and a variety of different 'temporal' evolutions are available (cf. Connes and Rovelli [1994]). These and other problems arising from the generality of the C^*-algebraic framework are avoided by the CBH reconstruction at the price of confining itself to the quantum informational setting.

Axiom CBH2 is used to establish that the C^*-algebras of Alice and Bob, $\mathcal{A}$ and $\mathcal{B}$, taken separately, are non-Abelian. Interestingly, non-Abelianness of $\mathcal{A}$ and $\mathcal{B}$ is proved by assuming that they are kinematically independent. This means that quantumness, of which non-Abelianness is a necessary ingredient, is not a property of any given system taken separately, as if it were the only physical system in the Universe. On the contrary, to be able to derive the quantum character of the theory, one must consider the system in the context of at least one other system that is physically distinct from the first one. As a consequence, *inter alia*, this forbids treating the whole Universe as a quantum system if one reconstructs quantum theory along the CBH lines.

Axiom CBH3 entails non-locality: spacelike separated systems must at least sometimes occupy entangled states. It is not proved, however, that actually instantiated states fill the space of *all* entangled states. CBH show that if Alice and Bob have spacelike separated quantum systems, but cannot prepare any entangled state, then Alice and Bob can devise an unconditionally secure bit commitment protocol. From this theorem the authors deduce that the impossibility of unconditionally secure bit commitment entails that 'if each of the pair of *separated* physical systems $\mathcal{A}$ and $\mathcal{B}$ has a non-uniquely decomposable mixed state, so that $\mathcal{A} \vee \mathcal{B}$ has a pair $\{\rho_0, \rho_1\}$ of distinct classically correlated states whose marginals relative to $\mathcal{A}$ and $\mathcal{B}$ are identical, then $\mathcal{A}$ and $\mathcal{B}$ must be able to occupy an entangled state that can be transformed to ρ_0 or ρ_1 at will by a local operation' (emphasis added). The term 'separated' is essential and, nevertheless, its precise meaning is not given. Once again, this can be compared to the confusion between *distinct* and *distant*. When CBH claim that Alice and Bob represent '*spacelike* separated systems,' while formally Alice and Bob are just two C^*-algebras, one sees how the way in which CBH apply the algebraic formalism is constrained by the context of quantum information theory. We witness here an interesting situation in which the language and the context used to formulate and to justify the fundamental principles set a limit on the applicability of the mathematical formalism used to represent these principles. Even if the formalism can be understood more generally than within the initially chosen disciplinary setting, one still cannot make his way out of this linguistic and contextual prison without running the risk of losing the meaning of the axioms. If one persists, however, and then obtains a new mathematical result, this result will be void of physical meaning until a new, broader justification of the fundamental principles is provided. Philosophical and linguistic justification, and mathematical derivation, play here a game of mutual onslaught and retreat which, ultimately, leads to the advance of science at the expense of what had previously been considered philosophy.

Notwithstanding the difficulties with justification, the CBH result would be a perfect example of reconstruction were it not for a great deal of mathematical structure which is implied by the choice of the C^*-algebraic framework. The assumptions of the algebraic formalism include at the very least, the relations between operators satisfying linearity, operators being subject to multiplication by one another, the number field being $\mathbb{C}$, and the states giving rise to the Hilbert space representation via the GNS construction. When one makes a complete list of all such tacit assumptions, the CBH reconstruction appears once again to suffer from the defect of incorporating a serious mathematical abstraction, similar to the derivations from axioms H1–H5 or R1–R2.

3.5 Intentionally incomplete reconstructions

All reconstructions we have discussed earlier share the goal of deriving, at the final end, the full-blown structure of quantum theory. But it is perhaps not accidental that none of them could fully succeed. Recently, a new type of information-theoretic reconstructions appeared, *intentionally* not aimed at deriving the whole quantum theoretic structure (Aaronson [2004], [2005]; Barrett [2007]; Hardy [unpublished (a)]; Popescu and Rohrlich [1994]; Smolin [2005]; Spekkens [2007]). Christened pejoratively by their own authors, these 'toy models,' 'fantasy quantum mechanics,' or 'quantum mechanics lite' employ a methodology of reconstruction that has been previously overlooked by scientists: it is now claimed as helpful to reconstruct, not the full version, but only a certain part of quantum theory. One builds, therefore, a new theory which is, from the very beginning, not intended to be *the* quantum theory; but this new theory allows nevertheless to better understand the structure of the 'true' quantum theory. To the same old ultimate aim of better understanding quantum mechanics toy models provide a variety of new promising insights.

The idea of modifying usual quantum theory is not new. Non-linear extensions of the Schrödinger equation have been explored by various authors (Bialynicki-Bela and Mycielski [1976]; Gisin [1990]; Weinberg [1989]), and comprehensive reviews of these attempts can be found in (Nattermann [1997]; Svetlichny [unpublished]). Non-linear models are also sometimes analyzed in the context of quantum theories more general than quantum mechanics, for example, in quantum gravity. Intentionally incomplete reconstructions, however, differ substantively from these attempts to modify quantum theory.

While the latter take standard linear quantum mechanics to be incomplete and purport to replace it by a non-linear, more complete theory, toy models see *themselves* as incomplete. They do not question the validity of quantum mechanics and do not compete with it in explaining empirical phenomena. As is routinely emphasized in the opening paragraphs of articles introducing toy models, (in Barrett [2007], for example), methodologically toy models focus on important physics principles, which are upheld while other possibilities are modified. This way the consequences of such and such principle are investigated, independently of other principles. Thus, intentionally incomplete reconstructions are not aimless, but allow one to achieve a better understanding of the structure of quantum theory. Incompleteness, then, becomes a *feature* rather than a flaw of toy models.

Most of the existing examples of toy models are based on information-theoretic principles. To compare the toy model with standard quantum theory, one then asks whether the former reproduces quantum computational phenomena available in the latter. Among others, such questions may include:

- Does the toy model allow superluminal signalling?
- Does the toy model allow bit commitment?
- Does the toy model allow teleportation, dense coding, or remote steering?
- Does the toy model allow exponential speed-up relative to classical computation or solving NP-complete problems in polynomial time?

Other toy models are inspired by information-theoretic principles, but they are compared with standard quantum theory in the aspects that do not necessarily relate to computation:

- Does the toy model allow non-locality and to what extent?
- Is the toy model contextual?
- Does the toy model possess a continuum of states, measurements and transformations between states?

Answers given to all of these questions can be yes or no depending on the model. Investigating then the difference between the first principles of a particular toy model, and of standard quantum mechanics, one learns with precision which fundamental principle is responsible for which element of the quantum theoretic structure. Spekkens's toy model, for example, accommodates such quantum phenomena as non-commutativity, interference, the multiplicity of convex decompositions of a mixed state, no cloning, teleportation, and others (Spekkens [2007]). We learn that the continuous state space, the existence of a Bell theorem, or contextuality, all of which are absent from this toy model, go unconnected with the appearance of the phenomena that are reproduced. Analogously, the toy model known as non-local, or PR, boxes (Popescu and Rohrlich [1994]; Barrett [2007]) allows non-local correlations that are strictly stronger than those allowed by quantum mechanics, while it only slightly modifies the quantum mechanical state space. One then sees that non-locality is not an exclusively quantum feature and, further, that the amount of non-locality in quantum theory is smaller than in some other theories. We can then conjecture, for instance, that the 'true' quantum theory has as much non-locality as it does, and not more or less, due to the continuum of states and to reversible transformations between pure states, both of these being left out of the PR boxes toy model.

These examples show how one obtains a deeper insight into the structure of quantum theory and therefore achieves what was initially promised by the program of intentionally incomplete reconstruction. Toy models, although incomplete, form a very fertile class of reconstructions. Their recent advent in the area of the foundations of quantum theory manifests the fact that the shift from interpretation of quantum theory to its reconstruction gave birth, in this area, to many a new, previously non-existent idea.

4 Conclusion

We have argued that reconstruction is the exclusive way to make things *clear* about quantum mechanics. As such, this idea is not novel but has been in the air for some time, and a concise statement can, for example, be found in (Rovelli [1996]),

> Quantum mechanics will cease to look puzzling only when we will be able to *derive* the formalism of the theory from a set of simple physical assertions ('postulates,' 'principles') about the world. Therefore, we should not try to append a reasonable interpretation to the quantum mechanical formalism, but rather to *derive* the formalism from a set of experimentally motivated postulates.

What is novel, however, is that an increasing number of researchers work nowadays on reconstructing quantum theory, and the time is ripe to promote this general framework to the status of a widely accepted paradigmatic shift in the domain of the foundations of physics.

Reconstruction brings in clarity to where interpretation was struggling to make sense of a physical theory. What belongs to physical theory is no more than what is needed for its derivation. All other questions belong to metatheory and are related to the justification task for the choice of first principles, conducted externally. Reconstruction is successfully competing with more traditional interpretations, due to its appealing conceptual transparency and to the clarity that it brings into the structure of the theory. However, completely reconstructing quantum theory remains only a partially solved problem, and intentionally incomplete reconstructions show that this apparent failure may not be accidental. Although to a varying degree, mathematical abstraction is a necessity for each of the currently existing reconstructions. Also, it would be too ambitious to expect that all of modern quantum theory, including field theory and quantum gravity, could be derived from a few axioms. Still, if one wishes to *understand* the meaning of even most advanced parts of quantum theory, and to reach a consensus in this understanding, it is then inevitable that simple physical principles be formulated and put in the very foundation of the reconstruction of quantum theory.

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